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Relativistic Hydrostatic Structure Equations and Analytic Multilayer Stellar Model

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Abstract

The relativistic extension of the classic stellar structure equations is investigated. It is pointed out that the Tolman-Oppenheimer-Volkov (TOV) equation with the gradient equation for gravitational mass can be made complete as a closed set of differential equations by adding that for the Tolman temperature, and the set is proposed as the relativistic hydrostatic structure equations. The exact forms of the relativistic Poisson equation and the steady-state heat conduction equation in the curved spacetime are derived. The application to an ideal gas of particles with the conserved particle number current leads to a strong prediction that the heat capacity ratio almost becomes one in any Newtonian convection zone such as the solar surface. The steady-state heat conduction equation is solved exactly in the system and thermodynamic observables exhibit the power law behavior, which implies the possibility for the system to be a new model of stellar corona and a flaw in the earlier one obtained by using the non-relativistic stellar structure equations. The mixture with another ideal gas yields multilayer structure to a stellar model, in which classic stellar structure equations are reproduced and analytic multilayer structure of luminous stars is revealed in a suitable approximation.

Keywords: Tolman temperature; Poisson equation; Tolman-Oppenheimer-Volkov

Introduction

As also read from descriptions in myths and Bible, the brightness of stars has been recognized as something indispensable and reverent from ancient time. The Sun has given energy to living beings during the day, shining stars have attracted people like jewels in the night. How they maintain energy and beauty has been a mystery before physicists unveil it. To reveal the mystery, Helmholtz and Kelvin proposed a hypothesis that the brightness of the Sun originated in the gravitational potential energy due to the contraction of solar fluid [1, 2]. See ref. for related history [3]. This proposal was rejected because only the gravitational contraction cannot produce sufficient energy to maintain the solar visibility longer than the time-scale of terrestrial minerals. Taking into account the progress of nuclear physics, Eddington pointed out that what is necessary for the Sun to gain sufficient luminous energy is certain subatomic reaction process, whose insight was subsequently developed further by Gamov and Bethe [4-6]. For a spherically symmetric star, this Eddington's insight is described by the following equation

$$\frac{dLr}{dr} = 4\pi r2\varrho\epsilon, \tag{1.1}$$

where Lr is the energy flowing outwards across a sphere of radius r called luminosity, ϱ the mass density, and ϵ the net energy

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production rate per unit mass liberated from all reaction processes including subatomic ones. If such produced energy is transported by radiation, then the temperature gradient inside a star is induced as [7].

$$\frac{dT}{dr} = -\frac{3\kappa\varrho}{16\sigma_B T^3}\frac{L_r}{4\pi r^2} \tag{1.2}$$

where$\boldsymbol{\sigma_B = \pi^2 c k_B^4/(60(c\hbar)^3}$ is the Stefan-Boltzmann constant and κ is the opacity of stellar fluid per unit mass, while if energy is transported by convection, the temperature gradient is replaced by

$$\frac{dT}{dr} = -(1-\frac{1}{\gamma})\frac{G_N M_{r\varrho} T}{r^2 P} \tag{1.3}$$

where γ is the specific heat ratio of stellar fluid, P the local pressure, GN the Newton constant, Mr defined by

$$\frac{dM_r}{dr} = 4\pi r^2 \varrho \tag{1.4}$$

On the other hand, the visibility of a luminous star is based on its stable existence, which is achieved as a result of the balance between the gravitational force and the internal pressure of stellar fluid. In a region where the gravitational force can be approximated by the Newtonian one, the hydrostatic equilibrium can be described by

$$\frac{dP}{dr} = -\frac{G_N M_r}{r^2}\varrho \tag{1.5}$$

The above set of differential equations is basic to investigate stellar structure and known as the stellar structure equations. Solving them combining equations of state for P, ϵ, κ in terms of ϱ, T with suitable boundary conditions, one can determine four macroscopic variables ϱ, T, Mr, Lr as functions of r. See ref.[8–13] .These differential equations also yield information on interdependent relations among global observables of luminous stars such as the mass-luminosity relation and the luminosity-temperature one depicted by the Hertzsprung-Russell diagram [5]. See ref. for their recent data [14-16]. The above traditional stellar structure equations have certainly played important roles to extract information on interior of luminous stars and interdependent relations of their global observables. However, they are apparently built on the basis of non-relativistic physics, which restricts them to be applicable only in the Newtonian regime such that density and pressure are sufficiently small like in the neighborhood of a stellar surface and the inside of a light star. See ref. [17]. Indeed, the relativistic extension of the equation for hydrostatic equilibrium (1.5) has already been investigated and established as the Tolman-Oppenheimer-Volkov (TOV) equation and the deviation between nonrelativistic results and relativistic ones increases as the mass density does [18-21]. Since there is an evolutionary process from a main sequence star to a highly dense one consisting of degenerate matter, it would be desirable to extend the stellar structure equations fully to their relativistic ones to be applicable in the non-Newtonian regime so as to be able to investigate the interior of a degenerate star and keep track of a stellar evolutionary process. With taking this into account, it is natural to ask how the traditional structure equations should be extended to the general relativistic ones, and whether any significant consequence can be drawn from this extension particularly to physics of luminous stars less dense than compact stars. (See textbooks, for instance on the study of a compact star using the TOV equation.) The purpose of this paper is to address these questions by employing some latest results of the author on relativistic local thermodynamics in relativistic hydrostatic equilibrium with spherical symmetry [22-24]. In the work, entropy current and entropy density were constructed by the proposed method in ref. [25]. The constructed entropy density was shown to satisfy the local Euler's relation and the first law of thermodynamics concurrently and non-perturbatively in the Newton constant, in which the local temperature is exactly coincident with the Tolman temperature [26]. On top of this, the established local thermodynamics was applied to such a hydrostatic equilibrium system with uniform energy density and the relativistic stellar structure was completely determined with the analytic expressions of all local thermodynamic observables. A lessen derived from this simple application is that the TOV equation and two gradient equations for gravitational mass and the Tolman temperature form a closed set of differential equations with one equation of state given. Based on these results, the author puts forward that the TOV equation and two gradient equations for the gravitational mass and the Tolman temperature are the relativistic structure equations for spherically symmetric hydrostatic equilibrium upgraded from the traditional stellar structure equations. A nontrivial point in this proposal is that the proposed relativistic structure equations do not contain variables on rate such as luminosity and energy production rate, so that they cannot be obtained simply by extending each traditional non-relativistic stellar structure equation to the relativistic one. Then the main issue of this paper is twofold: whether the proposed structure equations can reproduce the classic stellar structure equations leading to important stellar properties, and whether any new significant consequence can be drawn from the proposed relativistic ones properly. The goal of the paper is to answer these questions positively and provide evidence for validity and efficiency of the proposal. This is not trivial at all taking into account the fact that the proposed relativistic hydrostatic structure equations have less number of equations and that of variables as well. To the end, firstly, the setup of a relativistic hydrostatic equilibrium system with spherical symmetry is fixed and the necessary and sufficient set of the proposed structure equations is explicitly presented in section 2. It is pointed out that the proposed temperature gradient equation is consistent with a thermodynamic relation known in the ordinary thermodynamics and thus it is expected to hold for any local thermodynamic equilibrium system. Then the relativistic Poisson equation is derived nonperturbatively in the Newton constant, and it is converted into the differential equation for the Tolman temperature using its relation to the gravitational potential. This is the steady-state heat conduction equation exactly holding in this curved spacetime and plays a key role to determining the hydrostatic structure. In section 3, the proposed hydrostatic structure equations are applied to the construction of a model with multilayer structure of luminous stars including the Sun. Through this application, advantages of the relativistic extension of the structure equations become transparent. One of the advantages is seen in the temperature gradient equation. In the conventional stellar structure equations, the temperature gradient equation needs to be chosen as either (1.2) or (1.3) suitably in accordance with the way of energy transportation, or is newly computed by using the so-called mixing length theory [27, 28]. In the proposed ones, the temperature gradient equation is unchanged regardless of transport phenomena and the energy transport of fluid is described by its equation of state. In particular, it is shown that a simple fluid whose pressure is proportional to its energy density satisfies the conventional temperature gradient equation in a convective zone (1.3). Such a simple fluid can be realized by an ideal gas of particles with conserved particle number current, which is called baryonic particles. This model predicts that the heat capacity ratio is almost one in any Newtonian convection zone such as the solar surface. A characteristic feature of this new hydrostatic equilibrium model is that the steady-state heat conduction equation can be solved exactly and thermodynamic observables are determined non-perturbatively in the Newton constant. They exhibit the power law behavior with the power law index related to the heat capacity ratio and implies that this model can be used to describe an ionized state of fluid and a plasma one in stellar corona. An interesting feature in this model is that pressure is totally well-behaved and vanishing at the asymptotically far region. This result conflicts with the earlier one obtained by using the non-relativistic Newtonian gravity, and it concludes that the Newtonian approximation is not valid any more in stellar corona [29]. After applying to an ideal gas of non-relativistic particles to determine the local temperature by perturbation in section 3.2, an analytic stellar model is investigated as the hydrostatic equilibrium of two types of ideal gases of baryonic particles and non-relativistic ones in section 3.3. In the model, the ideal gas of baryonic particles forms a layer of the main stellar material and that of non-relativistic ones does a layer of atmosphere such as photosphere. In addition, by considering a situation with the system coupling to radiation, the luminosity can be related to the local temperature, which leads to the relativistic extension of the conventional temperature gradient equation in a radiation zone (1.2). A summary of the multi-layer stellar model is given in **TABLE 1**.

Table 1: A simple example of an analytic multi-layer stellar model is shown. In each layer, the local temperature and energy density determined in the main text are shown. In the column of 'Matter', the symbols 'NR', 'B', 'γ', and 'D' mean 'Non-Relativistic particles', 'Baryonic particles', 'photons', and 'Degenerate matter', respectively. The reason why photon is enclosed by bracket is that it is not involved in the stellar structure equations. $r_H = \frac{2G_N M_\star}{c^2}$, $r_c = \sqrt{\frac{3c^2}{8\pi G_N \rho_C}}$, $M_\star$ is the stellar gravitational mass, γ_b is the heat capacity ratio of the ideal gas of baryonic particles, while ν_* is given by (3.19), $\hat{n}_g$ (3.12), $T_\star$(3.41), $R_\star$(3.42), $T(R_*)$ (3.22), R_*(3.27), in the main text. It is possible to pile up another layer with boundary conditions connected smoothly.

Layer	Region	Temperature T	Energy density ρ	Matter
Empty	$R_\star \leq r \leq \infty$	$T_\star \sqrt{\frac{1-\frac{r_H}{R_\star}}{1-\frac{r_H}{r}}}$	0	(γ)
Atmosphere	$R_* \leq r \leq R_\star$	$T_\star (1 - \frac{G_N \nu_*}{2c^2} + \cdots$	$(\bar{m} c^2 + \frac{3}{2} k_B T)\, \hat{n}_g$	NR(+γ)
Interior	$R_c \leq r \leq R_*$	$T(R_*) \left(\frac{R_*}{r} \right)^{2\left(1-\frac{1}{\gamma b}\right)}$	$\rho(R_*) \left(\frac{R_*}{r} \right)^2$	NR+B(+γ)
Core	$0 < r \leq R_c$	$T(R_c) \frac{2\sqrt{r_c^2 - R_c^2}}{3\sqrt{r_c^2 - R_c^2} - \sqrt{r_c^2 - r^2}}$	ρ_c	D

Section 4 is devoted to summary and discussion including open problems and future works.

Relativistic hydrostatic structure equations

In order to generalize the hydrostatic structure equations so as to include the relativistic effect, a hydrostatic equilibrium system needs to be formulated to respect general covariance. Such a system with spherical symmetry was studied in [18, 26, 30, 31]. In which the line element generally takes the form such that

$$g_{\mu\nu} dx^\mu dx^\nu = -e^\nu \left(cdt \right)^2 + e^\lambda dr^2 + r^2 \left(d\theta^2 + \left(cos\theta \right)^2 d\phi^2 \right) \tag{2.1}$$

where ν, λ are functions of the radial coordinate r, and the energy-stress tensor is of the form of a perfect fluid

$$T_{\mu\nu} = \left(\rho + P \right) \frac{u_\mu}{c} \frac{u_\nu}{c} + P g_{\mu\nu}, \tag{2.2}$$

where ρ is local energy density, P is local pressure, u^μ is the four fluid velocity. Note that $c^2\nu/2$ corresponds to the gravitational potential probed by a point particle in the weak gravity regime, so ν itself will be also called the gravitational potential for convenience [23]. The balance between the gravitational attractive force and the repulsive one of fluid is described by the Einstein equation, which reduces in the comoving frame to

$$P' = -\frac{(P + \rho)}{2} \nu' \tag{2.3}$$

$$\lambda' = re^{\lambda} 8\pi G \frac{\rho}{c^2} - \frac{e^{\lambda} - 1}{r} \tag{2.4}$$

$$\nu' = re^{\lambda} 8\pi G \frac{P}{c^2} - \frac{e^{\lambda} - 1}{r} \tag{2.5}$$

where G:= GN/c^2. Eliminating λ and ν leads to the TOV equation [18].

$$P' = -\frac{(P+\rho)}{c^2} \frac{G(\check{E}_r + 4\pi r^3 P)}{r^2 (1 - 2\frac{G\check{E}_r}{rc^2}} \tag{2.6}$$

$$\check{E}_r' = 4\pi r^2 \rho \tag{2.7}$$

where $\check{\boldsymbol{E}}_{\boldsymbol{r}}$ is defined by $\check{\boldsymbol{E}}_{\boldsymbol{r}} := \frac{\boldsymbol{rc^2}}{\boldsymbol{2G}}(\mathbf{1} - \boldsymbol{e}^{-\boldsymbol{\lambda}})$ Note that $\check{\boldsymbol{E}}_{\boldsymbol{r}}$ has the dimension of energy. Employing the method given in one can construct entropy current and entropy density as a conserved current and a conserved charge density for this system [24, 25, 32]. A key to the construction is to find a vector field ξ^{μ} to satisfy a differential equation such that $\boldsymbol{T}^{\mu}_{\boldsymbol{\nu}} \boldsymbol{\nabla}_{\mu} \boldsymbol{\xi}^{\boldsymbol{\nu}} = \mathbf{0}$.

This vector field enables one to construct a conserved current as $\boldsymbol{J}^{\mu} = \sqrt{-\mathbf{g}} \boldsymbol{T}^{\mu}_{\boldsymbol{\nu}} \boldsymbol{\xi}^{\boldsymbol{\nu}}$ and a conserved charge as $\mathbf{Q} = \int \mathbf{d3xJt}$ in general. In order to find conserved entropy current for the fluid, we find such a vector field ξ^{μ} to be proportional to the fluid velocity u^{μ} . The entropy density constructed in this way was shown to satisfy the local Euler's relation and the first law of thermodynamics concurrently and non-perturbatively in the Newton constant in the comoving frame [24].

$$Ts = u + Pv, Ts' = u' + Pv' \tag{2.8}$$

Where $v = r^2 cos\theta$ $v = r^2 cos\theta$ $v = r^2 cos\theta \left(\frac{1 - 2G\check{E}_r}{rc^2} \right)^{-\frac{1}{2}}$, $u = \rho v$ is the internal energy density, $\boldsymbol{T} \propto \boldsymbol{e}^{\frac{-\boldsymbol{\nu}}{\mathbf{2}}}$ is the local temperature given by Tolman [26]. In particular, it was shown that the entropy density does not depend on the radial coordinate, which is a natural consequence for a steady state with vanishing heat flux in the energy stress tensor. Using (2.3), one can derive the temperature gradient equation as [24].

$$T' = \frac{P'}{P + \rho} T \tag{2.9}$$

Plugging (2.6) into this leads to

$$T' = -\frac{G\left(\check{E}_r + 4\pi r^3 P\right)}{c^2 r^2 \left(1 - 2\frac{G\check{E}_r}{rc^2}\right)} T \tag{2.10}$$

This temperature gradient equation (2.10) and the TOV equation (2.6) with (2.7) form a closed set of differential equations, and this set of three differential equations is proposed as the relativistic extension of classic stellar structure equations.

Comments are in order. Firstly, the temperature gradient equation (2.9) involves only thermodynamic variables and their gradients, which suggests that a corresponding relation exists in global thermodynamics on flat spacetime. The answer of such a thermodynamic relation is,

$$\overline{T}\left(\frac{\partial \overline{P}}{\partial \overline{T}}\right)_{\overline{V},\,\overline{N}} = \left(\frac{\partial \overline{U}}{\partial \overline{V}}\right)_{\overline{T},\,\overline{N}} + \overline{P}, \tag{2.11}$$

since this can be rewritten as $\left(\frac{\partial \overline{P}}{\partial \overline{T}}\right)_{\overline{V},\,\overline{N}} = \frac{\overline{T}}{\overline{P}+\overline{\rho}}$, where $\overline{\rho} = \left(\frac{\partial \overline{U}}{\partial \overline{V}}\right)_{\overline{T},\overline{N}}$ and the thermodynamic variables with overline stand for global quantities used here only [1]. It may not be so trivial to obtain the local thermodynamic relation (2.9) from (2.11), because the energy density ρ appearing in the energy stress tensor and the internal energy density u in the thermodynamic relations are different on curved spacetime, which could make it difficult to presume (2.9) from (2.11) without knowing the correct forms of the laws of local thermodynamics. Therefore the relation (2.9) is expected to hold for any local thermodynamic equilibrium system. Secondly, as is clear from the above derivation, there are several equivalent expressions for the temperature gradient equation. (2.10) may be useful for numerical analysis, while (2.9) or (2.3) may be more useful for non-numerical one. Which expression is best to be used will depend on context of analysis. Thirdly, although the constructed entropy density does not play any role in the structure equations, the existence thereof has important implications. One is that this ensures the validity of the definitions of the macroscopic quantities such as the temperature and the internal energy due to the fact that they satisfy the local Euler's relation and the first law of thermodynamics. Another is to assure the system to be in a local equilibrium, which justifies to solve the hydrostatic structure equations by assuming the distribution of an ideal statistical ensumble for local thermodynamic quantities. This point together with (2.9) will become important to include the temperature effect dynamically. In earlier study, the TOV equation (2.6) and (2.7) were used to determine the behavior of pressure and density by solving them combined with an equation of state with respect to pressure and density only but they can be used to determine the local temperature and include the thermal effect incorporating the temperature gradient equation (2.9) [23]. Finally, on the non-relativistic reduction, it is easily seen that the TOV equation with (2.7) reduce to the non-relativistic hydrostatic equation (1.5) with (1.4) in the Newtonian regime where

$$P \ll \rho, P \ll \frac{\check{E}_r}{4\pi r^3}, \frac{2G_N \check{E}_r}{c^2} \ll r \text{ so that } \rho \to \varrho c^2, \check{E}_r \to M_r c^2$$

On the other hand, it needs some preparation of setup to reproduce (1.2) and (1.3), so it will be confirmed later.

General relativistic Poisson equation and steady-state heat conduction equation

Before moving on to detailed applications of the proposed relativistic hydrostatic structure equations, it is convenient to investigate how to determine local temperature from them. From the definition of $\check{E}_r$ and (2.5), a concise expression of $\check{E}_r$ with respect to thermodynamic variables is derived as

$$\check{E}_r = \frac{\frac{c^2 r^2 v'}{G} - 8\pi r^3 P}{2(1+rv')} \tag{2.12}$$

with $v' = -2T'/T.^2$ Differentiating both sides with respect to r and using (2.3) and (2.7), one finds,

$$v'' + \frac{2v'}{r} + v'^2 - \frac{4\pi G}{c^2}((rv'+1)(rv'+2)\rho - (2r^2 v'' + rv'(rv'-3) - 6)P) = 0 \tag{2.13}$$

This can be rewritten into a general coordinate invariant form as,

$$\nabla^2 v = \frac{8\pi G}{c^2}(g^{\omega\mu} + 2u^{\omega}u^{\mu})\,T\,\omega\mu \tag{2.14}$$

which is the general relativistic extension of the Poisson equation. To prove this, first compute the Laplacian acting on the scalar field ν as,

$$\nabla^2 v = e^{-\lambda}(v'' + (\frac{2}{r} + \frac{v' - \lambda}{2})v')$$

$$= e^{-\lambda}v'' + \frac{v' + 4\pi Gr/c^2(P(rv'+3) - (rv'+1)\rho) + 2/r}{rv'+1}v' \tag{2.15}$$

where at the 1st equation the formula $\nabla^2 v = \frac{1}{\sqrt{|g|}}\partial_\omega \sqrt{|g|}g^{\omega\mu}\partial_\mu v$ was used and at the 2nd one the term containing λ ′ was computed by employing (2.4) and (2.5) as

$$e^{-\lambda}\frac{v' - \lambda'}{2} = 4\pi Gr\frac{P - \rho}{c^2} + \frac{2G\check{E}_r}{r^2c^2} \tag{2.16}$$

and (2.12) was used to substitute $\check{E}_r$. The coefficient of v” in (2.15) is $e^{-\lambda} = \frac{1+8\pi Gr^2P/c^2}{rv'+1}$ while that in (2.13) is $1 + \frac{8\pi Gr^2 P}{c^2} = e^{-\lambda}(rv' + 1)$. Thus multiplying rv′ + 1 for both sides in (2.13) and subtracting it from (2.15), one can remove ν ″ in (2.15) and the rest is [23].

$$\nabla^2 v = 8\pi G(\rho + 3P)/c^2 \tag{2.17}$$

This matches (2.14) and completes the proof. It is important to stress that the form of the relativistic Poisson equation (2.14) or (2.17) has been derived without using any approximation and holds non-perturbatively in the Newton constant.

The generalized relativistic Poisson equation can be solved order by order in the Newton constant. The leading order is

$$v'' + \frac{2v'}{r} + v'^2 = 0 \tag{2.18}$$

This has a trivial solution that ν is constant, while there is a non-trivial one as $v' = \frac{1}{r(-1+c_0 r)} =: v'_\star$ where C_o is an integration constants. This non-trivial solution corresponds to the exact one of the original differential equation (2.13) in the matter-empty region, P = ρ = 0. The integration constant C_0 is fixed by plugging this back into (2.12)

$$\check{E}_r = \frac{c^2 8\pi Gr^2(-1 + C_0 r)P}{2C_0 G} \tag{2.19}$$

Then evaluate this at a point r = $R_\star$ so far away from the core of the system that the contribution of the pressure to $\check{E}_r$ is negligible, $\check{E}_r \approx \frac{c^2}{2C_0 G}$ with r≥ $R_\star$ In this far region, $\check{E}_r$ is approximated as a constant and known as the gravitational mass by dividing it by c^2 , which is denoted by $M_\star$. Then the integration constant is fixed as $C_0 = \frac{1}{2M_* G}$ and $v_\star$ is given by $v_* = \log\left(\frac{1}{r} - \frac{1}{2M_* G}\right) + C_0$, where C_0 is another integration constant. This integration constant originates in the arbitrariness of the base point of the gravitational potential, which can be fixed by choosing an arbitrary referencing point. Here it is fixed by requesting it to vanish at the evaluation point: ν($R_\star$) = 0. Then the gravitational potential in the matter-empty region is determined as,

$$v_* = \log\left(\frac{1-\frac{2M_*G}{R_*}}{1-\frac{2M_*G}{r}}\right) \tag{2.20}$$

This exact gravitational potential in the matter-empty region is expanded with respect to the Newton constant as,

$$v_* = 2M_*G\left(-\frac{1}{r}+\frac{1}{R_*}\right)+\ldots \tag{2.21}$$

This leading term describes the Newton's law of universal gravitation for an object of the mass $M_\star$, which is the origin of the name of $M_\star$ mentioned above. As a result the evaluation point $r = R_\star$ should be chosen to satisfy the pressure at the point $P(R_\star)$ to satisfy an inequality

$$c^2 \gg 4\pi R_*^3 \frac{P(R_*)}{M_*} \tag{2.22}$$

with $R_\star \gg rH$ assumed. Accordingly the behavior of temperature in the matter-empty region is exactly determined as [24].

$$T = T(R_*)e^{-\frac{v}{2}} = T(R_*)\sqrt{\frac{1-\frac{r_H}{R_*}}{1-\frac{r_H}{r}}} =: T_{vac} \tag{2.23}$$

where $r_H := 2M_\star G$ is the radius of the horizon of the (Schwarzschild) black hole whose mass is identical to the stellar gravitational mass. The local temperature is formally divergent at the location of the horizon radius. It can be confirmed that (2.23) satisfies the differential equation $T_{vac}(T_{vac}^{''}+\frac{2}{r}T_{vac}^{'})-3T_{vac}^{'}{}^{2}=0$, which is identical to the leading order of the relativistic Poisson equation written in term of the local temperature as,

$$T(T''+\frac{2}{r}T')-3T'^2$$

$$=\frac{4\pi G}{c^2}(\rho(-2r^2T'^2+3rTT'-T^2)+P(4r^2T'^2-rT(2rT''-3T')-3T^2)) \tag{2.24}$$

This is the relativistic steady-state heat conduction equation for a relativistic hydrostatic equilibrium system with rotational symmetry. The next-to-leading order of the general relativistic Poisson equation in the Newton constant is determined by expanding the differential equation (2.13) as $\nu = v_0 + Gv_1 + O(G^2)$ with v_0 constant and taking the leading term as,

$$v_1^{''}+\frac{2}{r}v_1^{'}=\frac{8\pi}{c^2}(\rho_0+3P_0) \tag{2.25}$$

where ρ_0, P_0 are the leading order of the energy density and that of the pressure, respectively. In terms of the variable of temperature, the next-to-leading order of the relativistic Poisson equation is given by,

$$T_1'' + \frac{2}{r} T_1' = -\frac{4\pi T_0}{c^2}(\rho_0 + 3P_0), \tag{2.26}$$

where $T = T_0 + GT_1 + \cdots$ with T_0 constant. Inputing detailed information on fluid and solving the differential equation (2.25), (2.26) gives the subleading correction of the gravitational potential, the local temperature, respectively. Comments are given below. Firstly, a perturbative solution for (2.13) or (2.24) obtained this way is useful in a weak-gravity regime with energy density and pressure sufficiently small such as the vicinity of a stellar surface, while in a strong-gravity region such as the interior of a heavy stellar core, more useful information will be obtained by solving the original relativistic Poisson equation directly in a non-perturbative fashion with respect to the Newton constant. Secondly, astrophysical observation is basically done in a region far away from the core, so $r \gg r_H$. Therefore, the condition $r \gg 2G\hat{E}_r/c^2$ is assumed for practical results in what follows. In particular, this condition is satisfied in the Newtonian regime mentioned above.

Applications

In this section, the relativistic hydrostatic structure equations proposed in the previous section are applied to constructing a model of luminous stars. This application also demonstrates how classic results on a luminous star are reproduced or modified with taking into account the effect of general relativity and how to determine local thermodynamic observables inside a luminous star.

Ideal gas of baryonic particles

To the end of the construction of a model of luminous stars, first consider a locally hydrostatic equilibrium system of a single ideal gas of particles with conserved particle number current, which are called baryonic particles in this paper. The equation of state of an ideal gas is,

$$P = \hat{n} k_B T, \tag{3.1}$$

where $\hat{n}v$ gives the ordinary particle number density and k_B is the Boltzmann constant, while the conservation of the particle number current requires that $\nabla\mu J\mu = 0$, where $J\mu = \hat{n}u\mu$. The left hand side is computed in a radially moving frame as $\nabla\mu J\mu = ur\,(\hat{n}' + \hat{n}(\log(\sqrt{|g|}u^r))')$. A relativistic fluid equation derived in [24] rewrites this as $\nabla\mu J\mu = ur\,(\hat{n}' - \hat{n}\,\frac{\rho'}{\rho+P})$. Therefore the conservation of the number current leads to,

$$(\hat{n}' - \hat{n}\,\frac{\rho'}{\rho+P}) \quad (3.2)$$

Substituting (3.1) into this so as to eliminate ˆn and using (2.9), one finds $\frac{P'}{P} = \rho'/\rho$. This can be solved as

$$P = w\rho, \tag{3.3}$$

where w is an integration constant. The fluid for pressure to be proportional to energy density such as (3.3) is called a simple fluid in what follows for convenience. To fix the integration constant, recall that the energy density of an ideal gas is given by $u=nC_V T$, where C_V is the heat capacity at constant volume and $n = \hat{n}v$. This can be rewritten as $\rho = \hat{n}C_V$ T. By substituting this and (3.1) into (3.3), the integration constant can be fixed as,

$$w = \frac{k_B}{C_v} = \gamma - 1, \tag{3.4}$$

where $\gamma = C_P / C_V$ is the heat capacity ratio. Here C_P is the heat capacity at constant pressure and given by the Mayer's relation as $C_P = C_V + k_B$, which is guaranteed to hold by the laws of thermodynamics confirmed above. Plugging (3.3) into (2.9) with (3.4), one obtains,

$$\frac{T^{'}}{T}=\left(1-\frac{1}{\gamma}\right)\frac{P^{'}}{P}. \tag{3.5}$$

This is exactly the saturation point of the inequality of the Schwarzschild stability condition for a hydrostatic equilibrium system and known as an equation of state characteristic to a convection zone, in which the energy transport occurs mainly by convection. On the other hand, in order for fluid in hydrostatic equilibrium to satisfy (3.5), it needs to be a simple fluid satisfying (3.3) [33, 34]. As a result, the necessary and sufficient condition for energy to be transported by convection is that there is a proportionality relationship between pressure and energy density of fluid in the region. Note that the above equation (3.5) can be easily solved as $P\alpha T^{\frac{\gamma}{\gamma-1}}$. Thus $\rho = aT^{\frac{\gamma}{\gamma-1}}$, where a is a constant.

This new derivation of the temperature gradient equation in the convection zone leads to a new important prediction for a general property of stars. That is, in a region with the Newtonian approximation valid, pressure is negligible compared to energy density so that the parameter w almost vanishes [17]. Combining this with the observational fact that the Newtonian approximation is valid near the surface of most stars, one concludes that the heat capacity ratio in any stellar convection zone with the Newtonian approximation valid is almost one. This is consistent with a result in fluid dynamics on flat spacetime that an isentropic fluid satisfies a polytropic relation $P\alpha\rho^{\gamma}$.

$$\gamma - 1 \tag{3.6}$$

In particular, it is known that there exists considerably large convection zone underneath the solar surface [35, 36]. Therefore, it follows that fluid near the solar surface has the heat capacity ratio nearly equal to 1. This is a robust prediction newly made in this paper. The deviation of the heat capacity ratio from one will be described by stellar global observables as (3.28) later.

Power law behavior of thermodynamic observables

The radial dependence of temperature is determined by solving the relativistic steady-state heat conduction equation (2.24), which reduces to,

$$\mathrm{T}\left(T^{''}+\frac{2}{r}T^{'}\right)-3T^{'2}$$

$$=\frac{4\pi aG}{c^2}T^{\frac{\gamma}{\gamma-1}}\left(-6r^2T^{'2}+2r^2TT^{''}+2T^2+\gamma\left(4r^2T^{'2}-rT\left(2rT^{''}-3T^{'}\right)-3T^2\right)\right) \tag{3.7}$$

This can be solved perturbatively in the Newton constant as illustrated in section 2.1, while this expression implies the existence of an exact solution obeying the power law, $T=Ar^n$, where A is a nonzero constant and n is a negative number so that the temperature increases as it goes deeper into the center. These parameters are determined by substituting back into the differential equation, which simplifies to,

$$A^2\left(2n-1\right)r^{2n-2}\left(4\pi aGA^{\frac{\gamma}{\gamma-1}}(n(\gamma-2)+3\gamma-2)r^{\frac{n}{\gamma-1}+n+2}+c^2n\right)=0 \tag{3.8}$$

There exists a nontrivial solution if and only if $\frac{n}{\gamma-1}+n+2=4\pi aGA^{\frac{\gamma}{\gamma-1}}(n(\gamma-2)+3\gamma-2)r^{\frac{n}{\gamma-1}+n+2}+c^2n=0$. Thus the parameters for a nontrivial solution are determined as $n=-\frac{2(\gamma-1)}{\gamma}, A=(\frac{c^2(\gamma-1)}{2\pi aG(\gamma^2+4\gamma-4)})^{\frac{\gamma-1}{\gamma}}$. As a result, the relativistic steady-state heat conduction equation has been solved exactly as,

$$T=\left(\frac{c^2\left(\gamma-1\right)}{2\pi aG(\gamma^2+4\gamma-4)}\right)^{1-\frac{1}{\gamma}}\frac{1}{r^{2\left(1-\frac{1}{\gamma}\right)}}. \tag{3.9}$$

That is, the local temperature obeys the power law. Thus the density and the pressure do so as well, such that $P\propto\rho\propto 1/r^2$. Therefore it concludes that a hydrostatic equilibrium system of a simple fluid exhibits the power law behavior of macroscopic observables.

It is important to stress that this solution cannot be obtained by perturbation in the Newton constant, so that the power law behavior cannot be seen by solving the system perturbatively. It is also interesting that the parameters in this solution are given only by data of internal stellar fluid and not by external data such as the gravitational mass. From this result, the gravitational potential for this model can be also determined as $v=4\left(1-\frac{1}{\gamma}\right)\log\left(\frac{r}{R_\star}\right)$.

New model for stellar corona

The above results of the power law behavior of thermodynamic observables imply that this model is applicable to stellar corona, in which temperature is so much higher than the other part of stellar superficial region that gas is almost or completely ionized. This suggests that macroscopic behavior and transport phenomena in stellar corona can be investigated by combining plasma physics with stellar structure equations. For instance, temperature is predicted to fall off in a power law in solar corona. It was pointed out by Parker that combining the result of plasma physics for fully ionized gas with the conventional nonrelativistic hydrostatic structure equations leads to non-vanishing pressure at infinity [29]. He speculated on this as a signal of the impossibility for gas in the solar corona to reach hydrostatic equilibrium and presumed that the non-vanishing pressure at asymptotic region gives rise to gas streaming outward from the Sun called the solar wind, whose existence was suggested by Biermann earlier [37, 38]. The solar wind was indeed observed by a spacecraft launched some years later [39, 40].

This is a successful interplay between theory and experiment. However, it is also pointed out that there still remains unsolved issues for the corona and the solar wind by using the traditional stellar structure equations and results of plasma physics [41]. In the application of the above relativistic hydrostatic model of a simple fluid to stellar corona, however, there is no problem for the system to stay in hydrostatic equilibrium. Indeed, not only the local temperature but also the local pressure behave in the power law as shown above, which leads to pressure vanishing in the asymptotic region.

This result conflicts with the earlier one obtained by using the non-relativistic structure equations explained above. Then one might wonder which result is indeed correct. The answer of the author is that the earlier model contains a flaw in the argument. To explain this, one first needs to accept the fact that the relativistic hydrostatic structure equations presented in this paper, which are rigorously derived from the Einstein gravity, are more fundamental than the non-relativistic ones derived from the Newtonian gravity. Indeed, it has been confirmed that the relativistic hydrostatic structure equations reduce to the non-relativistic ones by neglecting terms due to the effect of general relativity. Such an approximation is valid only in the Newtonian regime, where the pressure is negligible compared to other terms. This implies that the Newtonian approximation cannot be justified for a solution such that the pressure is non-vanishing at a far distance. More concretely, one of the conditions to justify the non-relativistic reduction, $4\pi r^3 P \ll \breve{E} r$, is clearly violated for such a situation, and the contribution from pressure in (2.6) is not negligible at all. Parker derived the analytic expression of the number density for an ideal gas with temperature behaving as the power law by solving the non-relativistic hydrostatic equation (1.5), and found that it becomes divergent at infinity while, in the current system of an ideal gas with temperature obeying the power law, the number density has been obtained as [29].

$$\hat{n} = \frac{P}{k_B T} = \frac{a(\gamma - 1}{k_B}\left(\frac{c^2(\gamma-1)}{2\pi a G(\gamma^2 + 4\gamma - 4)}\right)^{\frac{1}{\gamma}} \frac{1}{r^{\frac{2}{\gamma}}} \tag{3.10}$$

which also obeys the power law and is vanishing in the asymptotic region. The latter fully contains general relativistic effect.

As a result it concludes that the Newtonian approximation is not valid for a hydrostatic equilibrium system in a plasma state such as stellar corona. Note that this conclusion does not deny the existence of the solar wind, whose existence must be accounted for by another mechanism. It is an interesting future work to study whether a newly proposed model compatible with relativistic structure equations can demonstrate any property on stellar corona and make prediction on it.

Comment on application to radiative fluid

One might wonder that this result is applicable also to an ideal gas of photon or the radiation dominant region, which is described by the equation of state (3.3) with $w = \frac{1}{3}$, so $P_{rad} = \frac{1}{3}\rho_{rad}\alpha T^4$, $\rho_{rad} = aT^4$. The integration constant a is fixed by considering a surface of a layer at which the Stefan-Boltzmann's law holds, so that a $= 4\sigma_B/$c. This layer may be called photosphere, which is conventionally specified by a single surface at a certain optical depth $\bar{\tau} = \frac{2}{3}$ defined by , $\bar{\tau} = \int_R^\infty k\varrho dr$ where R is a radius of a luminous star [12]. This leads to ρrad = 4σB c T 4 , which is consistent with the Bose distribution in equilibrium. Then the behavior of the local temperature would be given by (3.9) with the parameters specified above for the radiation dominant region. However, there is an important caveat for the application of the relativistic hydrostatic structure equations. That is, they were derived in the comoving observer with the perfect fluid, and such an observer does not exist for null fluid. Thus the application of the above results to radiation is not supported strictly speaking, though it may be useful merely as a rough estimation. Note that this caveat is also the case for the non-relativistic stellar structure equations. The proper analysis to include the effect of radiation to hydrostatic structure is left as future work.

Ideal gas of non-relativistic particles

Next, consider a system of an ideal gas consisting of non-relativistic particles for later use. The equation of state is given by P $= P_g$, $\rho = \rho_g$, where,

$$P_g = \hat{n}_g k_B T,\ \rho_g = (\bar{m}c^2 + \frac{3}{2}k_B T)\,\hat{n}_g, \tag{3.11}$$

where $\bar{m}$ is the averaged mass of the non-relativistic particles. This averaged mass is described as $\bar{m} = \mu_e m_u$, where μ_e is the mean molecular weight per free electron and m_u is the atomic mass unit defined by 1g/mol. Note that μe can be described by using the weight fraction of hydrogen denoted by X as $\mu_e = 2/(1 + X)$. Then consider to satisfy the temperature gradient equation (2.9), which is in the current setup given by $(P_g + \rho_g)T' = P_g' T$. Solving this differential equation, one can fix the form of the particle number density as,

$$\hat{n}_g = C_g T^{3/2} e^{-\frac{\bar{m}c^2}{k_B T}} \tag{3.12}$$

where C_g is an integration constant. This result is indeed consistent with statistical physics, and the integration constant can be fixed to match the Maxwell distribution as $C_g = \bar{g}(\frac{2\pi\bar{m}k_B}{h^2})^{\frac{3}{2}}$, where $\bar{g}$ corresponds to the mean internal degrees of freedom of a non-relativistic particle.

As previously, the radial dependence of temperature is determined by solving the steady-state heat conduction equation. Different from the previous case of simple fluid, it is very difficult to solve it exactly, so here it is solved up to the next-to-leading order. As shown in section 2.1, the leading order solution for (2.18) is a constant, ν_0 = const, so is the temperature, T_0 = const, and thus the above thermodynamic quantities as well:

$$P_0 = \hat{n}_{g0} k_B T_0, \;\; \rho_0 = (\bar{m}c^2 + \frac{3}{2} k_B T_0)\hat{n}_{g0,} \;\; \hat{n}_{g0} = \bar{g}(\frac{2\pi\bar{m}k_B}{h^2})^{\frac{3}{2}} T_0^{3/2} e^{-\frac{\bar{m}c^2}{k_B T_0}}. \tag{3.13}$$

Therefore the right-hand side of the next-to-leading order of the relativistic Poisson equation (2.25) becomes a constant, so it can be solved as,

$$a_b \approx \rho(R_*) / T(R_*)^{\frac{\gamma_b}{\gamma b - 1}} \tag{3.14}$$

where C_1, $\tilde{C}_1$ are integration constants. Plugging this back into (2.12) leads to,

$$\breve{E}_r = \frac{8\pi c^2 r^2 \rho_0 - 3c^4 C_1 r}{16\pi G r^3(\rho_0 + 3P_0) + 6c^2(r + C_1 G)} \tag{3.15}$$

Evaluating this at the surface of the inner layer, r= R_*, one finds,

$$c^2 M_* = \frac{8\pi c^2 R_*^4 \rho_0 - 3c^4 C_1 R_*}{16\pi G R_*^3(\rho_0 + 3P_0) + 6c^2(R_* + C_1 G)} \tag{3.16}$$

Where $M_* = \breve{E}_r(R_*)/c^2$. From this, the integration constant C_1 is fixed as,

$$C_1 = \frac{2R_* M_*(8\pi G R_*^2(\rho_0 + 3P_0)/c^2 + 3) - 8\pi R_*^4 \rho_0 / c^2}{3R_* - 6GM_*}$$

$$\approx 2M_* - \frac{8}{3}\pi R_*^2 \rho_0 / c^2, \tag{3.17}$$

while $\tilde{C}_1$ is determined to request ν_1 to vanish at r=$R_\star$ as introduced in section 2.1, outside which matter contribution is negligible so that (2.22) is satisfied:

$$\tilde{C}_1 = \frac{2M_*}{R_\star} - \frac{4\pi(3P_0 R_*^3 + \rho_0 R_*^3 + 2R_*^3)}{3c^2 R_\star} \tag{3.18}$$

Substituting this into (3.14) yields the solution of the gravitational potential at the subleading order as $\nu_1 \approx \nu_*$, where,

$$v_* = 2(\frac{1}{R_\star} - \frac{1}{r})\left(M_* + \frac{6\pi P_0 r R_\star (r + R_\star) + 2\pi\rho_0 (r^2 R_\star + rR^2 - 2R_*^3}{3c^2} \right) \quad (3.19)$$

This solution is valid in an annuls region with R* ≤ r ≤ R⋆ and should connect smoothly to the one in the matter-empty region (2.21) at r = R⋆, which imposes a condition to satisfy a relation between the stellar gravitational mass M⋆ and M* as,

$$M_\star = M_* + 4\pi R_\star^3 \frac{P_0}{c^2} + \frac{4}{3}\pi (R_\star^3 - R_*^3)\frac{\rho_0}{c^2} \quad (3.20)$$

at the leading order of the Newton constant. Then the local temperature is determined up to the next-to-leading order in G as,

$$T = T(R_\star)(1 - Gv_* / 2 + ...) \quad (3.21)$$

since T_0 = T(R⋆). In particular, evaluating at r= R*, one can compute the ratio of the temperatures at r=R⋆ and r=R* as,

$$\frac{T(R_*)}{T(R_\star)} = 1 + G(\frac{1}{R_*} - \frac{1}{R_\star})\left(M_* + \frac{6\pi P_0 R_* R_\star (R_* + R_\star) + 2\pi\rho_0 (R_*^2 R_\star + R_* R_\star^2 - 2R_*^3)}{3c^2} \right) \quad (3.22)$$

Note that the solutions (3.19) and (3.21) are applicable for a case with ρ_0, P_0 general constants.

For most observed stars, the thermal kinetic energy $k_B T$ is much smaller than the rest energy of the constituent particle $\bar{m}c^2$ near surface. For instance, the temperature of the Sun at the core is estimated as of order 107K≈1KeV/k_B, which is even smaller than the order of the electron mass. This means that if such a star would consist mainly of the ideal gas of non-relativistic particles, then it would collapse and not exist stably. Thus an ideal gas of non-relativistic particles will not be appropriate as main constituent material of stable stars.

3.3 Analytic multilayer stellar model

To the end of the construction of a model of a star with multilayer structure, finally consider the mixture of two ideal gases consisting of baryonic particles and non-relativistic ones as an example. The ideal gas of non-relativistic particles here plays a role of non-conserved particles such as ionized ones, while that of baryonic particles main stellar constituent material. The total pressure and the total energy density is given by,

$$\rho = \rho_b + \rho_g\,, \; P = P_b + P_g\,, \quad (3.23)$$

where the subscript b is used for baryonic component. The equation of state for an ideal gas of baryons is as usual given by,

$$P_b = \hat{n}_b k_B T, \quad (3.24)$$

while the baryonic particle number conservation implies,

$$\hat{n}_b' = \hat{n}_b \frac{\rho'}{\rho + P}, \quad (3.25)$$

as explained in section 3.1. Substituting (3.24) into (3.25) so as to remove ˆnb and employing (2.9), one finds $P_b'(P_g + \rho) = P_b(P_g + \rho)'$. This can be solved as $P_b = w_b(P_g + \rho)$, where w_b is an integration constant. ρ_b can be determined in terms of T by solving the temperature gradient equation (2.9).

For a realistic situation to a typical star, the local temperature away from the core is much smaller than the rest energy of the non-relativistic particle per the Boltzmann factor, T≪T_g, where $T_g := \frac{\bar{m}c^2}{k_B}$. In this situation, $\rho_g = \bar{m}c^2\hat{n}_g$, where $\hat{n}_g$ is given by (3.12), and $P_g \ll \rho_g$, so that $P = (1 + w_b)P_g + w_b\rho \approx w_b\rho$. This implies that in the realistic situation baryonic particles are dominant and the system can be approximately regarded as a simple fluid near and below the surface, so that the parameter wb is related to the heat capacity ratio of the stellar material γ_b as $w_b = \gamma_b - 1$, as investigated in section 3.1.5 The total energy density was determined to satisfy (2.9) as $\rho \approx a_b T^{\frac{\gamma_b}{\gamma_{b-1}}}$, where ab is an integration constant. To fix it, introduce the boundary of a layer of the stellar material whose radius is denoted by R_* and impose the energy density of main stellar constituent material to vanish at the boundary of the layer.

$$\rho_b(R_*) = 0 \tag{3.26}$$

This fixes the integration constant as $a_b \approx \rho(R_*)/T(R_*)^{\frac{\gamma_b}{\gamma b-1}}$, where $\rho(R_*) = \rho_g(R_*) \approx \bar{m}c^2\hat{n}_g(R_*)$ and $T(R_*)$ is the temperature at the surface of the lower layer related to the one at the surface of the upper layer as (3.22). On the other hand, the local temperature inside the star is approximated as the one to obey the power law as (3.9), $T \approx T(R_*)\left(\frac{R_*}{r}\right)^{2\left(1-\frac{1}{\Upsilon b}\right)}$ with

$T(R_*) = \left(\frac{c^2(\gamma_b - 1)}{2\pi a_b G(\gamma_b^2 + 4\gamma_b - 4)}\right)^{1-\frac{1}{\gamma_b}} \frac{1}{R_*^{2(1-\frac{1}{\gamma_b})}}$. Substituting the expression of a_b into this, the radius of the baryonic layer is determined as,

$$R_* \approx \sqrt{\frac{c^2(\gamma_b - 1)}{2\pi\rho(R_*)G(\gamma_b^2 + 4\gamma_b - 4}} \tag{3.27}$$

The physical implication of this result is twofold. One is that a layer with bigger density goes deeper inside the star, which is physically preferable and indeed observed in the onion structure of a star. The other is that from this expression the deviation of the heat capacity ratio from one can be computed as,

$$\gamma_b \approx 1 + 2\pi\frac{\rho(R_*)}{c^2}GR_*^2 \tag{3.28}$$

where $\gamma_b \sim 1$ was used. For the case of the Sun, it is evaluated as $\gamma_b - 1 \approx 1\times10^{-6}$, where the following data were used: $\rho(R_*)/c^2 \to m_u / \left(\frac{4}{3}\pi a_B^3\right) \approx 2.7\times10^3 kg/m^3$, $R_* \to \frac{R_\star}{2} = \frac{R_\odot}{2} = 3.5\times10^8 m$, where a_B is the Bohr radius, $a_B = 5.3\times10^{-11}$m. This rough estimation implies that the deviation of the heat capacity ratio from one is very small. The total energy density and the baryonic component thereof are determined as,

$$\rho \approx \rho(R_*)\left(\frac{R_*}{r}\right)^2, \quad \rho_b \approx \rho\times\left(1 - (\frac{R_*}{r})^{1-\frac{3}{\gamma_b}} e^{T_g(\frac{1}{T(R_*)} - \frac{1}{T})}\right) \tag{3.29}$$

Then the baryonic component of pressure is given by $P_b \approx w_b\rho$, and the baryonic number density is $\hat{n}_b = P_b/k_B T$.

These results of thermodynamic observables are valid in a parametric region with $r_H \ll r \leq R^*$, $T \ll T_g$. Indeed, these expressions of thermodynamic variables develop singularity at the center of the system. This implies the existence of a core layer around the center in which thermodynamic behavior changes from the power law. Such a core deep inside a star is expected to consist of degenerate matter, so it may be very roughly approximated as a hydrostatic equilibrium state with uniform energy density [22]. The pressure and temperature can be analytically computed as follows [24, 42],

$$P = \frac{\sqrt{r_c^2 - r^2} - \sqrt{r_c^2 - R_c^2}}{3\sqrt{r_c^2 - R_c^2} - \sqrt{r_c^2 - r^2}} \rho_c \, , \; T = \frac{2\sqrt{r_c^2 - R_c^2}}{3\sqrt{r_c^2 - R_c^2} - \sqrt{r_c^2 - r^2}} T(R_c), \tag{3.30}$$

where ρ_c is the constant energy density, R_c is the radius of the surface of the core, $r_c = \sqrt{\frac{3}{8\pi G \rho_c}}$.

Beyond the layer of main stellar material, r>R*, the existing stellar matter is only the ideal gas of non-relativistic particles. Therefore, thermodynamic observables in this region are already computed in the previous section 3.2. From these data of energy density, M_* and $M_\star$ are computed from (2.7) as,

$$M_* = \frac{1}{c^2} \int_0^{R_*} dr 4\pi r^2 \rho, \; M_\star = M_* + \frac{1}{c^2} \int_{R_*}^{R_\star} dr 4\pi r^2 \rho \, . \tag{3.31}$$

The difference of these masses can be estimated as (3.20). If the core layer is negligibly small and the energy density can be always approximated by (3.29) inside the star,then M_* is evaluated as,

$$M_* \approx \frac{\rho(R_*)}{c^2} 4\pi R_*^3 \tag{3.32}$$

This has a simple physical meaning that the total mass of a star can be estimated just by the product of mass density at the surface of the baryonic layer and its volume up to a numerical factor 3. To confirm validity of this expression, below estimate the solar mass by using it. To this end, assume that the contribution of the atmosphere to the mass is negligible, $M_\star \approx M_*$, and that the radius of the upper layer is set to be identical to the solar one related to that of the lower layer as $R_\star \equiv R_\odot = \eta R_*$, where η is a numerical factor greater than 1. On the other hand, the mass density at the surface of the baryonic layer is estimated as $\frac{\rho(R_*)}{c^2} = \bar{m}\hat{n}_g(R_*) = \mu_e m_u / (\frac{4}{3}\pi a_B)^3$, where the number density is $\hat{n}_g(R_*) \approx 1/(\frac{4}{3}\pi a_B)^3$ with a_B the Bohr radius and $\mu_e \approx 1.2$ since the weight fraction of hydrogen is X = 0.6 for the case of the Sun [43]. This approximation means that the Sun is treated as a dense pack of hydrogens.

From these data, the solar gravitational mass is evaluated as,

$$M_\star \approx \frac{\mu_e m_u}{\frac{4}{3}\pi a_B^3} \times 4\pi (\frac{R_\odot}{\eta})^3 \approx \frac{1}{n^3} \times 10^{34} g. \tag{3.33}$$

This estimation yields the same order of the solar mass, $M_\odot = 2.0 \times 10^{33}$g.

In order to construct a model of a luminous star, consider energy transported by radiation measured by the luminosity L_r. In this paper, the luminosity is literally defined as the energy of radiation only flowing outwards across the sphere of radius r per unit time here. In some contexts, the luminosity is defined as the net energy flow transported not only by radiation but also by convection in a more involved context such as the mixing length theory. The reason to adopt the definition in this paper is because the net energy flow will be evaluated by using entropy as TdS. This is evaluated in the background metric (2.1) as

$$L_r = 4\pi r^2 e^{\lambda/2} F, \tag{3.34}$$

where F is the radiative energy flux summed all over the frequencies $F = \int dvF_v$ and F_v is the energy flux component of radiation with frequency v. Now assume that the source of radiation and the absorption body are absent in a region considered as observed in stellar surface. Then the energy flux component F_v is related to its pressure P_v by $\frac{dP_v}{d\tau_v} = \frac{F_v}{c}$, where τ_v is the frequency-sensitive optical depth defined through the absorption coefficient α_v as $d\tau_v = -\alpha_v dr$. Therefore $\frac{dP_v}{d\tau_v} = -\alpha_v \frac{F_v}{c}$. Summing all over the modes labeled by the frequency v leads to,

$$P_{rad}^{'} = -\alpha_R \frac{F}{c}, \tag{3.35}$$

where α_R is an averaged absorption coefficient called the Rosseland mean. The radiative component of pressure is given by $P_{rad}^{'} = \frac{4\sigma_B}{3c} T^4$, so $T^{'} = \frac{3c}{16\sigma_B T^3} P_{rad}^{'}$. Substituting (3.35) into this yields,

$$T^{'} = -\frac{3\alpha_R}{16\sigma_B T^3} F \tag{3.36}$$

Then using (3.34) one finds the relation between the temperature gradient and the local luminosity as,

$$T^{'} = -\frac{3\alpha_R}{16\sigma_B T^3} \frac{L_r}{4\pi r^2 (1 - \frac{2GE_r}{rc^2})^{-\frac{1}{2}}} \tag{3.37}$$

This is the relativistic extension of the temperature gradient equation (1.2) for the radiation dominant zone. As a byproduct, one can derive a relativistic result on the luminosity by plugging (2.10) into (3.37) as

$$L_r = \frac{64\pi\sigma_B}{3c^2\alpha_R} \frac{G(\check{E}_r + 4\pi r^3 P)}{(1 - 2\frac{G\check{E}_r}{rc^2})^{3/2}} T^4. \tag{3.38}$$

Indeed, near the stellar surface, (3.37) reduces to (1.2) with taking into account the relation between the Rosseland mean and the opacity as $\alpha R \approx \varrho K$ with ϱ the mass density of stellar matter, as asserted before, while (3.38) reduces to

$$L_r \approx \frac{64\pi\sigma_B}{3c^2\alpha_R} G\check{E}_r T^4. \tag{3.39}$$

Note that these expressions for the luminosity are valid in the region with matter present. In the matter empty region, the Rosseland mean and opacity vanish, and from (3.36), the temperature is constant. This is consistent with the result of the local temperature in the matter empty region (2.23) effectively. To extract physical implication, evaluate (3.39) in the atmospheric layer, $R_* \le r \le R_\star$. Near the upper surface, $r \lesssim R_\star$, the pressure is negligible from (2.22), and E_r and T are almost constant from (2.12) and (2.23), respectively, as seen in section 2.1. Thus

$$L_r \approx \frac{64\pi\sigma_B}{3\alpha_{R_\star}} GM_\star T(R_\star)^4 \tag{3.40}$$

Here α R⋆ is the Rosseland mean near the upper surface and becomes a constant. This shows that the luminosity in this region is also constant, which is consistent with the Stefan-Boltzmann law, $Lr \approx L_\star$, where

$$L_\star = 4\pi R^2 \sigma_B T_\star^4 \tag{3.41}$$

Here the radius $R_\star$, which was introduced so as to satisfy the condition (2.22), is de_ned for the Stefan-Boltzmann law to hold at the radius, and $T_\star$ is the effective surface temperature identified with $T_\star = T(R_\star)$. Then a novel relation is obtained as

$$R_\star^2 = \frac{16GM_\star}{3\alpha_{R\star}}. \tag{3.42}$$

In order to verify this expression, one needs to estimate the Rosseland mean given by the product of the mass density and the opacity, both of which are local quantities depending on measured position. Here, in order to do a rough estimation, evaluate the Rosseland mean and the gravitational mass by using the average mass density $\bar{\varrho}$ in the atmosphere as

$$\mathrm{M}_\star \approx \bar{\varrho}\frac{4}{3}\pi R_\star^3,\ \ \alpha R\star \approx \bar{\varrho}\mathcal{K}_\star \tag{3.43}$$

Then (3.42) can be rewritten as

$$R_\star \approx \frac{9\mathcal{K}_\star}{64\pi G_N} \tag{3.44}$$

In order to estimate the opacity in the atmosphere $\mathcal{K}_\star$, assume that the temperature in the atmosphere is so high that electrons are almost ionized to behave as free particles. This is indeed expected in solar corona. Then the opacity is estimated as $\mathcal{K}_\star \approx \sigma_T / \bar{m}$ where σ_T is the cross-section of the Thomson scattering, σ_T = 6:7 × $10^{-29}m^2$ [12, 13, 43]. Using this approximation, (3.44) is rewritten as

$$R_\star \approx \frac{9\sigma_T}{64\pi G_{N\bar{m}}} \tag{3.45}$$

For the case of the Sun, this is estimated as $R_\star \approx 3\times10^{-7}$m, which may not be so bad in comparison to the observed value, $R_\odot = 7 \times 10^8$m. It would be interesting to do precise evaluation by using detailed data of the opacity [12, 44, 45].

Analytic multilayer solar model

In order to build a concrete model of a star from the above multilayer stellar model, internal data of the star is necessary. To illustrate how to build such a concrete model, here construct a multilayer model of the Sun as an example. It is known that the Sun has a convection zone up to around three quarters of the total radius from the surface and the deeper rest is a radiation zone except the core [35, 36]. Here, for simplicity, the region of solar corona is neglected for simplicity, and the convective zone and the radiative one are both realized by a baryonic layer consisting of an ideal gas of baryonic particles with the heat capacity ratio varying in the layer. Since the total solar mass is already estimated correctly as seen in (3.33), below investigate behaviors of local observables

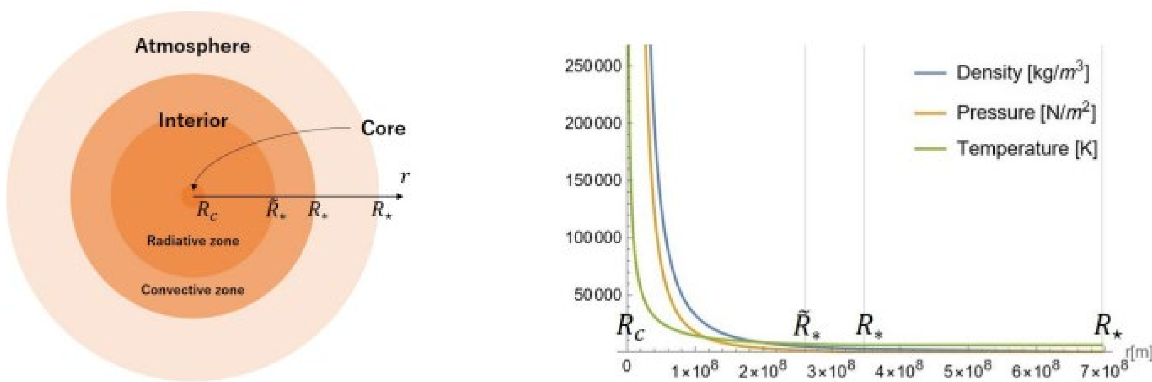
The surface of the atmosphere starts from the radius R⋆, which is set to the solar one, $R_\star := R_\odot$ .The layer of atmosphere continues roughly up to the half of the solar radius, $R_\star := \frac{1}{2}R_\star$ from which the baryonic layer starts. The baryonic layer is roughly divided into a convective zone, which ends at the radius $\tilde{R}_* = \frac{3}{4}R_*$ and a radiative one, which ends at r = R_c and connects to the core smoothly.

Here the radius of the core is chosen as $R_c := 5r_H$ by hand for illustration. Accordingly the heat capacity ratio starts with 1 at the surface of the baryonic layer and increases as it goes deeper. It takes the value 4/3 at the transition point from the convective zone to the radiative zone, and 5/3 at the one from the baryonic layer to the core, since the core is expected to be supported by repulsive force of degenerate electron gas whose heat capacity ratio is 5/3 [22, 46].

The behaviors of local observables inside the Sun are summarized in table 1 and figure 1. Their characteristic values are easily calculated as follows. On the local temperature, as an initial condition, $T_\star$ is set to the solar surface temperature observed by an asymptotic observer in accord with the Stefan-Boltzmann law, $T_\star := T_\odot \simeq 6\times10^3$K. The temperature does not change so much in the layer of atmosphere, so that it lasts at the surface of the baryonic layer, $T(R_*) \simeq T_\odot$. Then it increases as it goes deeper. At the transition point from the convective zone to the radiative one,

$T\left(\tilde{R}_*\right)=T\left(R_*\right)\left(\frac{R_*}{\tilde{R}_*}\right)^{1/2}\approx 7\times10^3 K$, while at the point from the baryonic layer to the core

$$T\left(R_c\right)=T\left(R_*\right)\left(\frac{R_*}{R_c}\right)^{1/2}\approx 2\times10^7 K .$$



(a) Solar multilayer structure. (b) Behaviors of local observables.

Figure 1: **(a) A schematic picture of solar multilayer structure and (b) the behaviors of local thermodynamic observables are drawn. Three local observables are plotted in the same graph even though they have different units in order to describe their behaviors concisely**

Note that in the current case, $r_c \gg R_c$, so the temperature of the center of the system is almost equal to the one at the surface of the core, $T(0) \approx T(R_c)$. The core temperature estimated here is of the same order as the earlier one obtained by using the classic stellar structure equation [35, 47].

On the density, an initial condition is given at the boundary between the atmosphere and the interior, $r = R_*$, where the density is estimated as a dense pack of hydrogens $\rho\left(R_*\right)/c^2 \approx m_u / \left(\frac{4}{3}\pi a_B^3\right) \approx 3\times10^3 kg/m^3$.

Then at the transition point from the convective zone to the radiative one $\rho\left(\tilde{R}_*\right)/c^2 = \rho\left(R_*\right)/c^2\left(\frac{R_*}{\tilde{R}_*}\right)^2 \approx 5\times10^3 kg/m^3$, while at the core $\rho_c/c^2 = \rho\left(R_*\right)/c^2\left(\frac{R_*}{R_c}\right)^2 \approx 1\times10^{12} kg/m^3$.

This density at the core is much bigger than the earlier result around 1.5 × 10^5kg/m^3 [35, 47]. In fact, the value is still bigger than the known average density of a typical white dwarf around 3 × 10^9kg/m^3 but less than that of a neutron star around 5 × 10^{14}kg/m^3 [42]. However, this result of high density is in some sense reasonable because the core is assumed to be supported by repulsive force of degenerate electrons so its density is expected to be comparable to that of a white dwarf. Accordingly the pressure in the model also becomes much higher than the earlier result.

The behaviors of local observables for this solar model are plotted in figure 1 (b) by determining the heat capacity ratio in each zone by the linear interpolation using the boundary values. They are qualitatively different from the earlier ones [35, 47]. It is important to stress that the values obtained here are very rough without fine tuning. Therefore they can be possibly made more precise by tuning the parameter with more precise data. Such a precise evaluation and further discussion of physical interpretations of these differences are left to future work.

Comment on the Eddington bound

In the above model of luminous stars, the radiation pressure is assumed to be too small to involve hydrostatic structure [44]. This approximated treatment is not used in the mixing length theory, in which both the convective energy flux and the radiant energy flux involve the traditional hydrostatic structure equations [27, 28, 48]. This approximation will be valid as long as the outward force caused by radiation is sufficiently small compared to the gravitational one produced by matter inside the star:

$$\left|\frac{dP_{rad}}{dr}\right| \ll \left|\frac{dP}{dr}\right|. \tag{3.46}$$

Using (3.34) and (3.38), one can compute the left-hand side as $\left|P'_{rad}\right| = \frac{\alpha_R L_r}{4\pi r^2(1-2\frac{G\tilde{E}_r}{rc^2})^{-\frac{1}{2c}}}$ while the right-hand is already computed as (2.6). Therefore the above inequality becomes

$$L_r \ll (1+\omega_b)\rho\frac{4\pi G(\check{E}_r+4\pi r^3 P)}{\alpha_{R^C}(1-2\frac{G\check{E}_r}{rc^2})^{\frac{3}{2}}}, \tag{3.47}$$

where $P \approx w_b\rho$ was used. Further estimation of the right-hand side needs to specify the expression of the Rosseland mean. In a non-relativistic hydrostatic system, the Rosseland mean takes the form of the product of the mass density ϱ and the opacity κ. In a general relativistic hydrostatic system, the energy density ρ is more fundamental than the mass density ϱ, so the Rosseland mean is defined by using the energy density as $\alpha_R = \kappa\rho/c^2$. Then the above can be rewritten as

$$L_r \approx \gamma_b\frac{4\pi G(\check{E}_r+4\pi r^3 P)}{\kappa(1-2\frac{G\check{E}_r}{rc^2})^{\frac{3}{2}}}. \tag{3.48}$$

The right-hand side corresponds to the relativistic extension of the Eddington limit for this stellar model. Indeed, evaluating this around a stellar surface with the Newtonian approximation valid, one finds

$$L_{R_\star} \ll \frac{4\pi c g_N M_\star}{\kappa} \quad (3.49)$$

DISCUSSION

The relativistic extension of the classic stellar structure equations has been investigated in the Einstein gravity. It has been pointed out that the TOV equation and two gradient ones for gravitational mass and Tolman temperature form a closed set of differential equations, and the set has been put forward as the desired relativistic extension of the stellar structure equations. The proposed relativistic hydrostatic structure equations have been shown to be endorsed by laws of local thermodynamics.11 From them, the exact form of the relativistic Poisson equation has been derived, and it has been converted into the steady-state heat conduction equation holding non-perturbatively with respect to the Newton constant by using the relation between the gravitational potential and the Tolman temperature. A couple of applications have been presented. One is to a hydrostatic equilibrium system consisting of a single ideal gas of particles with the particle number current conserved called baryonic particles. In this system, the proposed temperature gradient equation reduces precisely to the conventional one known in the convection zone. It has been predicted that the heat capacity ratio almost becomes one in the Newtonian convective regime such as the neighborhood of the solar surface. The generalized steady-state heat conduction equation has been solved exactly and thermodynamic observables determined non-perturbatively in the Newton constant, which exhibit the power law behavior such that $T \propto r^{\frac{-2(\gamma-1)}{\gamma}}, \rho \propto p \propto r^{-2}$ with γ the heat capacity ratio. This result implies that this model is applicable to a plasma state and thus to stellar corona. Another application is to a hydrostatic equilibrium system of an ideal gas of non-relativistic particles, for which the local temperature has been determined perturbatively in the Newton constant. Finally, by combining two ideal gases of baryonic particles and non-relativistic ones, an analytic multilayer structure of luminous stars has been presented as a simple example. In this model, the convective zone consists mainly of the ideal gas of baryonic particles, while the atmosphere is constituted by non-relativistic particles. This model also admits the layer of degenerate core and the stellar corona by changing the constituents suitably. By coupling the system to radiation, the traditional temperature gradient equation in a radiation zone has been extended to its relativistic version. In the proposed hydrostatic structure equations, the relativistic extension for the luminosity gradient (1.1) is not included. This is simply because structure equations are closed without the variable of luminosity. This implies that the variables on rate are not fundamental for relativistic hydrostatic equilibrium. This conclusion is actually preferable with taking into account the existence of a non-radiating star which can be described by a relativistic hydrostatic equilibrium system. However, this does not mean that the gradient equation for luminosity is not useful in a relativistic hydrostatic equilibrium system. Its naive relativistic extension will be given by

$$\frac{dL_r}{d_r} = \frac{4\pi r^2}{(1-2\frac{G\check{E}_r}{rc^2})^{\frac{1}{2}}}\rho\varepsilon, \quad (4.1)$$

which could be useful to extract information of the radiant energy production rate ϵ per unit energy for a hydrostatic stellar system with radiation. The information of energy production rate becomes important to build a model in detail in particular to investigate an evolutionary process of a star as a non-equilibrium open system including dissipation [35] . Further extension including such a hydrodynamical process is an interesting future work. An important application of the relativistic hydrostatic structure equations with a simple fluid model has been to stellar corona, in which pressure vanishes at the far asymptotic region. This result conflicts with the earlier one obtained by using the non-relativistic stellar structure equations, and what is a flaw in the earlier argument is ignoring a term containing general relativistic effect in the stellar equilibrium equation which cannot be neglected in a situation with pressure finite in the far asymptotic region. It would be interesting to investigate this model further to solve unsolved problems known in solar corona [35, 36]. In the multilayer stellar model presented in this paper, it consists of two ideals gases of baryonic particles and non-relativistic ones with radiation probed as a simple example. It is also possible to pile up a similar layer or another one consisting of a different type of fluid. Such a multilayer structure should be arranged to build a model for each observed star. In the analysis, radiation does not involve in the structure equations but just plays a role of a probe of local temperature. This approximated treatment of radiation is justified as long as the Eddington bound is sufficiently satisfied and convenient to analyze the stellar system near the stellar surface. However the radiative contribution may not be negligible inside the star, in particular, with so high temperature that an effective method of analysis is only numerical calculation. It is important to extend the presented relativistic hydrostatic structure equations so as to include the radiative contribution. In this paper, the proposed relativistic hydrostatic structure equations have been applied to the construction of a model of luminous stars. It would be tempted to apply them to a different type of stars such as a compact star consisting of degenerate matter and to investigate the finite temperature effect. A caveat for the application of the relativistic hydrostatic structure equations to such compact stars is to include the local chemical potential and temperature so as to be consistent with the local thermodynamic relations. It would be of great interest to investigate how the thermodynamic relations shown in including chemical potential in such a system is derived and whether such thermal interaction gives rise to any significant effect on the system [24]. The author hopes to address these issues and will report the progress in the near future.

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